Isogeometric Analysis, Symbol Approach, and Structured Matrices:

from the Spectral Analysis

to the Design of Fast Iterative Solvers

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Model problem

$$\begin{cases} -\Delta u + \boldsymbol{\beta} \cdot \nabla u + \gamma u = f & \text{on } \Omega = (0,1)^d \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

where $d \geq 1, \quad f \in L^2(\Omega), \quad \boldsymbol{\beta} = (\beta_1, \ldots, \beta_d) \in \mathbb{R}^d, \quad \gamma \geq 0$

Weak form

Find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = F(v) \qquad \forall v \in H_0^1(\Omega) \tag{2}$$

where $a(u,v) = \int_\Omega (\nabla u \cdot \nabla v + \boldsymbol{\beta} \cdot \nabla u \; v + \gamma u \, v), \qquad F(v) = \int_\Omega f \, v$

$\exists!$ solution $u \in H_0^1(\Omega)$ to (2), called the weak solution of (1)

Galerkin method

1. Choose a subspace $W \subset H_0^1(\Omega)$ with $\dim W = N < \infty$
2. Find $\tilde{u} \in W$ such that

$$a(\tilde{u}, v) = F(v) \qquad \forall v \in W \tag{3}$$

Whatever W, $\exists!$ solution $\tilde{u} \in W$ to (3), which is taken as an approximation to u

Chosen a basis $\{\varphi_1, \dots, \varphi_N\}$ for W, problem (3) is equivalent to solving the linear system

$$A\mathbf{u} = \mathbf{f}$$

where $A = [a(\varphi_j, \varphi_i)]_{i,j=1}^N$ is the **stiffness matrix** and $\mathbf{f} = [F(\varphi_i)]_{i=1}^N$

In our IgA setting W is chosen as a space of splines

We start with $d = 1$:

- $W = W_n^{[p]}$ = space of splines of degree p on the uniform mesh $\frac{i}{n}$, $i = 0, \ldots, n$, vanishing at $x = 0, 1$
- basis of $W_n^{[p]}$ = **B-spline basis**

For $d \geq 2$:

$$W = \underbrace{W_n^{[p]} \otimes W_n^{[p]} \otimes \cdots \otimes W_n^{[p]}}_{d \text{ copies}}$$

= space generated by the **tensor-product B-splines** vanishing on $\partial\Omega$

$A_n^{[p]}$ = stiffness matrix resulting from these choices

- **Asymptotic spectral distribution** of the sequence of (normalized) IgA matrices

$$\{n^{d-2} A_n^{[p]}\}_n$$

$$\Downarrow$$

Target: **find out the symbol** of the IgA matrices (**Weyl sense**)

Definition: spectral distribution of a sequence of matrices $\{X_n\}$ – symbol

Let

- $\{X_n\}$ = sequence of matrices, X_n of size $d_n \to \infty$
- $f : D \subset \mathbb{R}^m \to \mathbb{C}$ measurable function, $0 < \text{measure}(D) < \infty$

$\{X_n\}$ is **distributed like** f **in the sense of the eigenvalues**, in symbols $\{X_n\} \sim_\lambda f$, if

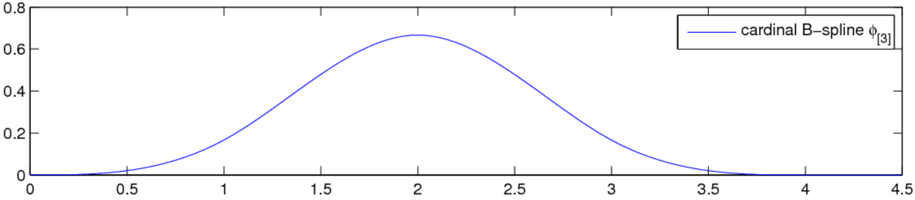
$$\lim_{n\to\infty} \frac{1}{d_n} \sum_{j=1}^{d_n} F(\lambda_j(X_n)) = \frac{1}{\text{measure}(D)} \int_D F(f(\mathbf{x}))d\mathbf{x} \qquad \forall F \in C_c(\mathbb{C})$$

- f = **symbol** of $\{X_n\}$

Informal meaning of $\{X_n\} \sim_\lambda f$

If f is smooth, then the eigenvalues of X_n behave as a uniform sampling of f over D

$\phi_{[p]}$ = **cardinal B-spline** of degree p on the uniform knot sequence $0, 1, \dots, p+1$



For $p \geq 1$, consider these functions over $[-\pi, \pi]$:

$$h_{p-1}(\theta) = \phi_{[2p-1]}(p) + 2\sum_{k=1}^{p-1} \phi_{[2p-1]}(p-k)\cos(k\theta), \qquad f_p(\theta) = (2 - 2\cos\theta)h_{p-1}(\theta)$$

If $d = 1$

Theorem

$$\{n^{-1}A_n^{[p]}\}_n \sim_\lambda f_p$$

The theorem holds for every d with $d-2$ in place of -1 and with

$$f_p : [-\pi, \pi]^d \to \mathbb{R}$$

$$f_p(\theta_1, \ldots, \theta_d) = \sum_{k=1}^{d} h_p(\theta_1) \cdots h_p(\theta_{k-1}) f_p(\theta_k) h_p(\theta_{k+1}) \cdots h_p(\theta_d)$$

$$f_p(\theta) = (2 - 2\cos\theta) h_{p-1}(\theta)$$

If $d = 1$

Theorem

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$$f_p(\theta) = (2 - 2\cos\theta) h_{p-1}(\theta)$$

For $d = 1$ the symbol is $f_p(\theta) = (2 - 2\cos\theta)h_{p-1}(\theta)$

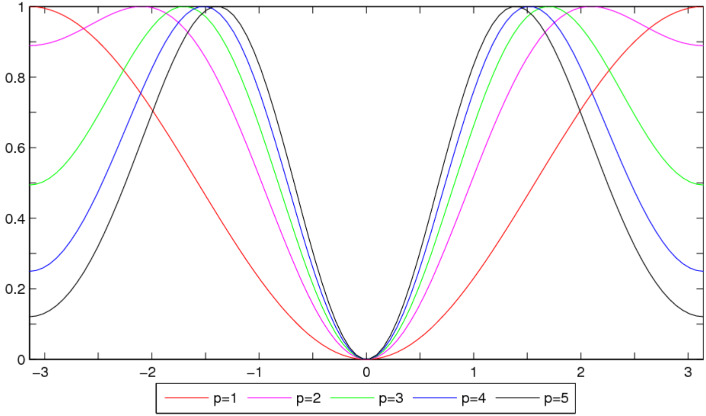


Figure: graph of the normalized symbol f_p/M_{f_p}

- $\lim_{\theta \to 0} \frac{f_p(\theta)}{\theta^2} = 1, \quad f_p(\theta) > 0$ for $\theta \neq 0 \Rightarrow \theta = 0$ unique zero of f_p with order 2
- setting $M_{f_p} = \max_{\theta} f_p(\theta), \quad \frac{f_p(\pi)}{M_{f_p}} \leq \frac{f_p(\pi)}{f_p\left(\frac{\pi}{2}\right)} = \frac{1}{2^{p-2}} \to 0$ exponentially

The normalized symbol f_p/M_{f_p} has a a numerical zero at $\theta = \pi$ for large p!

Besides the canonical zero $\theta = 0$, when p is large the normalized symbol has a non-canonical numerical zero at $\theta = \pi$

In the d-variate case, the situation is even worse!

Besides the canonical zero $(\theta_1, \ldots, \theta_d) = (0, \ldots, 0)$, when p is large the normalized symbol has **infinitely many** non-canonical numerical zeros located at the π-edge points

$$\{(\theta_1, \ldots, \theta_d) : \theta_j = \pi \text{ for some } j\}$$

From the properties of the symbol:

- standard multigrid methods for the IgA matrices, which take care of the actual zero $(\theta_1, \ldots, \theta_d) = (0, \ldots, 0)$, will be optimal, i.e. with convergence rate independent of n
- for large p, standard multigrid methods, which do not take care of the numerical zeros at the π-edge points, will have a bad convergence rate

$$\Downarrow$$

multi-iterative idea (S. , Comput. Math. Appl. 1993) to be fully considered for designing optimal and robust solvers

Target: **use carefully the symbol** to design fast (optimal and robust) multi-iterative solvers for the IgA matrices (alternative direction by Sangalli, Tani)

Standard two-grid methods for solving linear systems with matrix $n^{d-2}A_n^{[p]}$ involve:

- a standard coarse-grid correction with **full-weighting** projector
- a post-smoothing iteration by the standard (relaxed) **Gauss-Seidel** method

Symbol interpretation

Full-weighting treats properly the unique zero $(\theta_1, \ldots, \theta_d) = (0, \ldots, 0)$ of the symbol...
but both full-weighting and Gauss-Seidel ignore the numerical zeros arising for large p

... and, indeed, the standard two-grid method is optimal, but not robust...

n	$p=1$	$p=3$	$p=5$
16	0.16	0.64	0.96
28	0.17	0.64	0.96
40	0.18	0.64	0.96
52	0.18	0.65	0.96
n	$p=2$	$p=4$	$p=6$
17	0.27	0.88	0.99
29	0.27	0.88	0.99
41	0.29	0.88	0.99
53	0.30	0.88	0.99

Table: spectral radius

Multi-iterative idea: keep the full-weighting projector for dealing with the zero $(\theta_1, \ldots, \theta_d) = (0, \ldots, 0)$ and replace the Gauss-Seidel smoother with another smoother that takes care of the numerical zeros of the symbol

A suitable smoother is suggested by the symbol f_p itself:

take as smoother the PCG or the PGMRES with preconditioner having itself a symbol s_p which "deletes" the numerical zeros of our symbol f_p, yielding a p-independent preconditioned symbol $s_p^{-1} f_p$

In 1D:

$$f_p(\theta) = (2 - 2\cos\theta) h_{p-1}(\theta)$$

$\Rightarrow$ the **preconditioned symbol** $[h_{p-1}(\theta)]^{-1} f_p(\theta) = 2 - 2\cos\theta$ is p-**independent**

The idea can be generalized to the d-dimensional setting.

For the smoother PCG / PGMRES use a preconditioner with symbol

$$s_p(\theta_1, \ldots, \theta_d) = h_{p-1}(\theta_1)h_{p-1}(\theta_2)\cdots h_{p-1}(\theta_d)$$

Possible choice: Toeplitz matrix generated by s_p

Remark

Since the symbol s_p of the preconditioner is a **separable trigonometric polynomial**, a linear system associated with the preconditioner is easily solvable

Here we consider the system

$$n^{d-2}A_n^{[p]}\mathbf{u} = \mathbf{b}$$

coming from the IgA approximation of (1) in the case $d = 2,\ \beta = 0,\ \gamma = 1,\ f = 1$

n	$p=1$		$p=3$		$p=5$	
40	6	7	6	14	6	54
60	6	7	6	14	6	49
80	5	7	6	13	6	46
100	5	7	6	13	6	44
120	5	7	6	13	6	42
n	$p=2$		$p=4$		$p=6$	
41	6	8	6	29	6	115
61	6	8	6	27	5	104
81	6	9	6	26	5	97
101	6	9	6	26	5	91
121	6	9	6	25	5	87

Table: number of iterations

Here we consider the system

$$n^{d-2}A_n^{[p]}\mathbf{u} = \mathbf{b}$$

coming from the IgA approximation of (1) in the case $d = 3,\ \beta = 0,\ \gamma = 1,\ f = 1$

n	$p = 1$	$p = 3$	$p = 5$
14	6	6	8
24	6	6	7
34	6	6	7
44	6	6	6
n	$p = 2$	$p = 4$	$p = 6$
15	8	6	7
25	7	6	7
35	7	6	6
45	6	6	6

Table: number of iterations

Here we consider the system

$$n^{d-2} A_n^{[p]} \mathbf{u} = \mathbf{b}$$

coming from the IgA approximation of (1) in the case $d = 2$, $\boldsymbol{\beta} = \mathbf{0}$, $\gamma = 0$, $f = 1$

For the solution: V-cycle and W-cycle multigrid

n	$p = 1$		n	$p = 3$		n	$p = 5$	
16	10	7	14	7	6	12	7	7
32	11	7	30	9	6	28	8	6
64	12	7	62	9	6	60	10	6
128	13	7	126	10	6	124	11	6
256	13	7	254	11	6	252	12	6
n	$p = 2$		n	$p = 4$		n	$p = 6$	
15	8	6	13	7	6	11	7	7
31	9	6	29	8	6	27	8	6
63	10	6	61	10	6	59	10	6
127	11	6	125	11	6	123	11	6
255	12	7	253	12	6	251	12	6

Table: number of iterations

Here we consider the system

$$n^{d-2} A_n^{[p]} \mathbf{u} = \mathbf{b}$$

coming from the IgA approximation of (1) in the case $d = 3,\ \boldsymbol{\beta} = \mathbf{0},\ \gamma = 0,\ f = 1$

For the solution: V-cycle and W-cycle multigrid

n	$p = 1$		n	$p = 3$		n	$p = 5$	
16	10	7	14	7	6	12	8	8
32	11	7	30	8	6	28	8	7
64	12	7	62	9	6	60	9	6
n	$p = 2$		n	$p = 4$		n	$p = 6$	
15	9	8	13	7	6	11	9	9
31	8	7	29	8	6	27	8	6
63	9	7	61	9	6	59	10	6

Table: number of iterations

We have obtained **optimal and robust multi-iterative multigrid methods**

We have: fast multi-iterative solver for $A_n^{[p]}$ = Parametric Laplacian matrix (PL-matrix)

- "Parametric": the considered domain is the hypercube $(0,1)^d$
- "Laplacian": the considered problem is $-\Delta u = f$

What about IgA matrices $\mathcal{A}_n^{[p]}$ associated with full elliptic problems over general Ω?

The PL-matrix $A_n^{[p]}$ is an optimal and robust CG/GMRES preconditioner for $\mathcal{A}_n^{[p]}$

$$\begin{cases} -\nabla \cdot K\nabla u + \boldsymbol{\beta} \cdot \nabla u + \gamma u = f & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{4}$$

with Ω being a quarter of annulus:

$$\Omega = \{(x, y) \in \mathbb{R}^2 : \ r^2 < x^2 + y^2 < R^2, \ x > 0, \ y > 0\} \qquad r = 1 \qquad R = 4$$

and

$$K(x, y) = \begin{bmatrix} (2 + \cos x)(1 + y) & \cos(x + y)\sin(x + y) \\ \cos(x + y)\sin(x + y) & (2 + \sin y)(1 + x) \end{bmatrix}$$

$$\boldsymbol{\beta}(x, y) = \sqrt{x^2 + y^2} \begin{bmatrix} \cos \frac{x}{\sqrt{x^2+y^2}} \\ \sin \frac{y}{\sqrt{x^2+y^2}} \end{bmatrix}$$

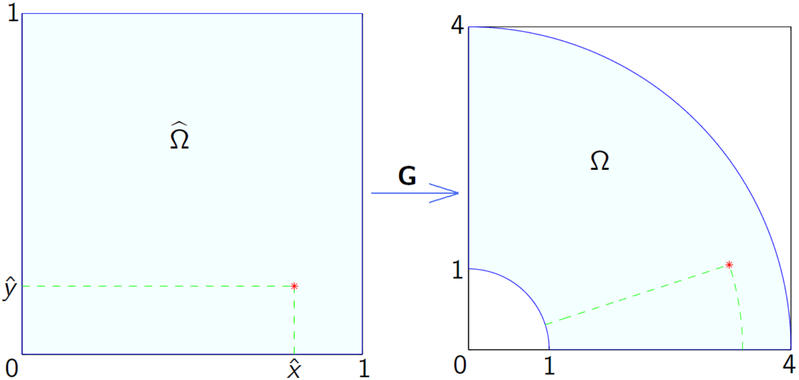
$$\gamma(x, y) = xy$$

$$f(x, y) = x \cos y + y \sin x$$

Isogeometric approach:

- take $\mathbf{G} : \widehat{\Omega} := (0,1)^2 \to \overline{\Omega}$ that describes Ω exactly:

$$\mathbf{G} : \widehat{\Omega} \to \overline{\Omega} \qquad \mathbf{G}(\hat{x},\hat{y}) = (x,y) \qquad \begin{cases} x = [r + \hat{x}(R-r)]\cos(\frac{\pi}{2}\hat{y}) \\ y = [r + \hat{x}(R-r)]\sin(\frac{\pi}{2}\hat{y}) \end{cases}$$



- approximate (4) with the Galerkin method:
  - approximation space: $\mathcal{W} = (W_n^{[p]} \otimes W_n^{[p]}) \circ \mathbf{G}^{-1}$
  - basis functions: **G**-deformations of the **tensor-product B-splines** defined on $\widehat{\Omega}$

$\mathcal{A}_n^{[p]}$ = resulting stiffness matrix

n	$p=1$		$p=2$		$p=3$		$p=4$		$p=5$		$p=6$	
10	29	16	25	18	42	19	72	21	119	22	164	23
20	61	20	42	21	50	22	84	23	140	24	223	25
30	94	22	63	23	60	23	90	24	154	25	240	26
40	128	23	84	24	77	24	95	25	161	26	249	26
50	161	24	106	24	96	25	106	26	168	26	256	27

Table: number of iterations

- The symbol can be recovered in the IgA Collocation/Galerkin setting with variable coefficient PDEs, general physical domain, general geometrical mapping
- The symbol can be recovered in the FEM setting with variable coefficient PDEs, general physical domain, general graded griddings
- Concerning the numerical methods, the dimensionality d is not an issue and singular mappings are not an issue
- We are now completing the analysis when the model space is given by NURBS

* The symbol approach is useful for understanding the spectral features of the IgA matrices and for designing efficient iterative solvers

* The symbol approach is not limited to IgA approximation techniques: it is a general tool for dealing with all local approximation techniques for PDEs, such as FD methods, FE methods, collocation methods, etc

* As done in this presentation, we identify **two steps in the symbol approach:**
  - **find out the symbol** for the specific approximation technique under consideration and study its properties
  - **use the symbol** and its properties to design efficient iterative solvers of Krylov, multigrid, or multi-iterative type

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THANK YOU VERY MUCH FOR YOUR ATTENTION

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